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ON THE RANGE OF ACCRETIVE OPERATORS[†]

BY

MICHAEL G. CRANDALL[‡] AND AMNON PAZY

ABSTRACT

The solvability of the nonlinear operator equation $w = x + Bx$, where B is accretive in a general Banach space X is studied by means of discrete approximations In particular, if B is continuous and everywhere defined an algorithm is given for solving the equation.

Introduction

Let X be a real Banach space and let $B: D(B) \subseteq X \to X$ be a nonlinear operator. B is called *accretive* if

$$\|x_1 - x_2\| \leqq \|x_1 - x_2 + \lambda(Bx_1 - Bx_2)\| \quad \text{for} \quad \lambda > 0, \qquad x_1, x_2 \in D(B).$$

If B is accretive, then B is *m-accretive* if $X = R(I + B)$, i.e. for every $w \in X$ there is an $x \in D(B)$ such that $w = x + Bx$. One of the first results in the study of accretive operators was obtained by G. Minty [5] and implied that every continuous everywhere defined accretive operator in a Hilbert space is m-accretive. This latter result was extended to general Banach spaces by R. Martin [4]. The known proofs of Martin's theorem employ the solvability of the initial-value problem

$$\text{(1)} \qquad \begin{cases} \dfrac{du}{dt} + Au = 0 \\ \\ u(0) = x \end{cases}$$

where A is continuous and accretive on X. The existence theory for equation (1) has been generalized to allow cases in which A is neither continuous nor single-valued. For recent developments see, e.g., Y. Kobayashi [3] and M. Pierre

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[7, 8]. This existence theory is rather technical and complex. The present paper was motivated by the desire to find a proof of the above mentioned theorem of Martin which is direct, constructive and which does not rely on the solvability of (1).

If B is continuous, everywhere defined and accretive we show in Theorem 2 of Section 1 how to choose numbers $\alpha_k \in (0,1]$ so that the sequence defined by $x_{k+1} = \alpha_{k+1}x_k - (1-\alpha_{k+1})(Bx_k - z)$ converges to the unique solution x_∞ of $x_\infty + Bx_\infty = z$. (The choice of α_{k+1} depends on x_k and $\alpha_1 + \cdots + \alpha_k$.) In fact, our main results easily adapt to provide elementary proofs (i.e., proofs not relying on (1)) of the strong generalizations of Martin's theorem obtained in [3] and [7]. In particular, the perturbation theorem of Webb [9] as generalized by Barbu [1] follows easily.

The main results are stated in Section 1 and proved in Section 2. Then various known results are obtained as applications in the final Section 3.

1. The main results

Let B be a mapping from X to the subsets of X which is accretive, i.e.

$$\|x_1 - x_2 + \lambda(y_1 - y_2)\| \geqq \|x_1 - x_2\| \tag{1.1}$$

for $\lambda > 0$, $y_i \in Bx_i$, $x_i \in D(B) = \{x \in X : Bx \neq \emptyset\}$. Given $w \in X$ we consider the solvability of the problem $w \in R(I+B)$, i.e. can we find $x \in X$ such that $w \in x + Bx$ (equivalently, $w - x \in Bx$). Replacing B by B_w where $B_w x = Bx - w$ we reduce to the case $w = 0$.

DEFINITION. A sequence $\{x_k\}_{k=0}^{\infty}$ is admissible for the problem $0 \in R(I+B)$ if there exist $y_k \in Bx_k$ and numbers $h_k > 0$, $k = 1, 2, \cdots$ such that

$$\begin{cases} \text{(i)} & \sum_1^\infty h_k = \infty, \\ \text{(ii)} & \sum_1^\infty \|x_{k+1} - x_k + h_{k+1}(x_{k+1} + y_{k+1})\| < \infty. \end{cases} \tag{1.2}$$

The first result is:

THEOREM 1. *Let B be accretive and $\{x_k\}$ be an admissible sequence for $0 \in R(I+B)$. Then*

(a) $$x_\infty = \lim_{k\to\infty} x_k \textit{ exists.}$$

Moreover

(b) $(1+\lambda)\|x-x_\infty\| \leqq \|x-x_\infty+\lambda(y+x)\|$ *for* $\lambda>0$, $x\in D(B)$, $y\in Bx$.

We expect x_∞ of (a) above to solve $0\in x_\infty+Bx_\infty$, but cannot in fact state so without further assumptions on B. (As we will see, (b) is close to the assertion $0\in x_\infty+Bx_\infty$.) Theorem 1 leaves open the question of existence of admissible sequences, which also requires further conditions on B (see Theorem 3 and Proposition 1 below). These questions are resolved for continuous B in the next result.

THEOREM 2. *Let $B: D(B)\subseteq X\to X$ be continuous and accretive.*

(a) *Let $D(B)=X$. Given $x_0\in X$ inductively define*

$$(1.3)\qquad \begin{cases} h_{k+1}=2^{-n_k}, & k=0,1,\cdots \\ x_{k+1}=\dfrac{1}{1+h_{k+1}}x_k-\dfrac{h_{k+1}}{1+h_{k+1}}Bx_k, & k=0,1,\cdots \end{cases}$$

where n_k is the least nonnegative integer n such that

$$(1.4)\qquad \left\|B\left(\frac{2^n}{1+2^n}x_k-\frac{1}{1+2^n}Bx_k\right)-Bx_k\right\|<\exp\{-(h_1+h_2+\cdots+h_k+1)\}.$$

Then $\{x_k\}$ is admissible for $0\in R(I+B)$.

(b) *Let $\{x_k\}$ be admissible for $0\in R(I+B)$ and $D(B)$ be closed. Then*

$$x_k+Bx_k\to x_\infty+Bx_\infty=0$$

where $x_\infty=\lim_{k\to\infty}x_k$.

REMARKS. The scheme (1.3), (1.4) gives an algorithm for computing $\{x_k\}$. Our proofs will give explicit estimates on $\|x_k-x_\infty\|$, but we will not be able to estimate n_k (equivalently, h_k) in general and therefore the number of steps required to achieve a preassigned accuracy. There are many alternatives to (1.3), (1.4). In particular, we mention that $\exp\{-(h_1+h_2+\cdots+h_k+1)\}$ on the right of (1.4) can be replaced by $\psi(h_1+\cdots+h_k)$ for any decreasing integrable function $\psi:(0,\infty)\to(0,\infty)$.

If B is not necessarily continuous, we will use the condition

$$(\mathrm{R})\qquad \inf_{\lambda>0}\frac{d(R(I+\lambda(I+B)),x)}{\min(1,\lambda)}=0\quad\text{for}\quad x\in\overline{D(B)},$$

where $d(C,x)$ denotes the distance from $C\subseteq X$ to $x\in X$.

THEOREM 3. *Let B be accretive and* (R) *hold. Then*:

(a) *There is an admissible sequence $\{x_k\}$ for $0 \in R(I+B)$.*

(b) $0 \in \overline{R(I+B)}$ *(the closure of $R(I+B)$). In particular, if B is also closed, there is a solution x_∞ of $0 \in x_\infty + Bx_\infty$ to which every admissible sequence converges.*

REMARKS. Suppose we knew that there is a subset D_0 of $\overline{D(B)}$ with the following property: For every $\varepsilon > 0$ and $x_0 \in D_0$ there are finite sequences $\{x_1, x_2, \cdots, x_N\} \subseteq D_0$, $\{h_1, \cdots, h_N\} \subset (0, \infty)$ and $y_i \in Bx_i$ such that $2 \geqq \Sigma_{j=1}^N h_j \geqq 1$ and

$$\sum_{i=0}^{N} \| x_{i+1} - x_i + h_{i+1}(x_{i+1} + y_{i+1}) \| < \varepsilon.$$

Then there will be admissible sequences for $0 \in R(I+B)$, as is easy to see. Various conditions which imply this property are discussed by M. Pierre [7, 8]. We have singled out (R) here because of its simplicity and the fact that it falls outside the scope of previous works. In particular, the requirement that $\liminf_{\lambda \downarrow 0} \lambda^{-1} d(R(I + \lambda(I+B)), x) = 0$ for $x \in \overline{D(B)}$ (which allows use of the results of Y. Kobayashi [3]) implies (R) but is not implied by it. However, none of the above mentioned conditions are necessary for the existence of admissible sequences. We note:

PROPOSITION 1. *Let B be accretive and $0 \in \overline{R(I+B)}$. Then there is an admissible sequence for $0 \in R(I+B)$. Conversely, if there is an admissible sequence for $0 \in R(I+B)$ then 0 is in the closed convex hull of $R(I+B)$.*

REMARKS. (i) For exposition's sake we have discussed the problem $0 \in R(I+B)$ where B is accretive. All our results adapt at once to the case $0 \in R(A)$ when $A - \omega I$ is accretive for some $\omega > 0$.

(ii) It would be nice to know if the existence of an admissible sequence for $0 \in R(I+B)$ implies $0 \in \overline{R(I+B)}$.

Section 2

We begin with three lemmas which contain the heart of the proofs of Theorem 1–3. Lemmas 1 and 2 (when $A = 0$) are variations of estimates used by Kobayashi [3], (see also [2]). Theorems 1 and 3 are proved following the lemmas. The proof of Theorem 2, which is closely related to that of Theorem 3, is given next. The section ends with the proof of Proposition 1.

Hereafter B is accretive, $x_k \in X$, $k = 0, 1, \cdots$ and $y_k \in Bx_k$, $k = 1, 2, \cdots$ are regarded as given. Let $\{h_k\}_{k=1}^{\infty}$ be a sequence of real positive numbers and define $z_k \in X$ for $k = 1, 2, \cdots$ by

$$x_{k+1} - x_k + h_{k+1}(x_{k+1} + y_{k+1}) = h_{k+1} z_{k+1} \qquad k = 0, 1, 2, \cdots. \tag{2.1}$$

LEMMA 1. *Let $i > l > 0$ and (2.1) hold. Then*

$$\| x_i - x_l \| \leqq b_{i,l} \| x_l + y_l \| + \sum_{k=l+1}^{i} h_k \| z_k \| \tag{2.2}$$

where $b_{i,l} = \min(1, \Sigma_{k=l+1}^{i} h_k)$.

PROOF. From (2.1) we have

$$\begin{aligned} h_{k+1}\| z_{k+1} \| + h_{k+1} \| x_l + y_l \| &\geqq \| (1 + h_{k+1})(x_{k+1} - x_l) - (x_k - x_l) + h_{k+1}(y_{k+1} - y_l) \| \\ &\geqq \| (1 + h_{k+1})(x_{k+1} - x_l) + h_{k+1}(y_{k+1} - y_l) \| - \| x_k - x_l \| \\ &\geqq (1 + h_{k+1}) \| x_{k+1} - x_l \| - \| x_k - x_l \| \end{aligned}$$

where the first and second estimates are the triangle inequality and the third follows from the accretiveness of B. Thus

$$\| x_{k+1} - x_l \| \leqq (1 + h_{k+1})^{-1} \| x_k - x_l \| + h_{k+1}(1 + h_{k+1})^{-1}(\| x_l + y_l \| + \| z_{k+1} \|). \tag{2.3}$$

and iterating this from $k = l$ to $k = i - 1$,

$$\begin{aligned} \| x_i - x_l \| &\leqq \sum_{j=l+1}^{i} \prod_{m=j}^{i} (1 + h_m)^{-1} h_j (\| x_l + y_l \| + \| z_j \|) \\ &\leqq M_{i,l} \| x_l + y_l \| + \sum_{j=l+1}^{i} h_j \| z_j \|, \end{aligned} \tag{2.4}$$

where

$$M_{i,l} = \sum_{j=l+1}^{i} \prod_{m=j}^{i} (1 + h_m)^{-1} h_j \leqq \sum_{j=l+1}^{i} h_j. \tag{2.5}$$

Now

$$M_{i,l} \text{ is nondecreasing in } i \geqq l + 1 \text{ and } 0 \leqq M_{i,l} \leqq 1 \tag{2.6}$$

since

$$M_{i+1,l} = \frac{1}{(1 + h_{i+1})} M_{i,l} + \frac{h_{i+1}}{1 + h_{i+1}}$$

is a convex combination of 1 and $M_{i,l}$. Thus (2.2) follows from (2.4) because $M_{i,l} \leqq b_{i,l}$ by (2.5) and (2.6).

LEMMA 2. *Let* (2.1) *be satisfied,* $A, B \geqq 0$, $l > 0$ *and assume that*

$$\|x_i - x_l\| \leqq A + B \sum_{k=l+1}^{i} h_k + \sum_{k=l+1}^{i} h_k \|z_k\|$$

for $i \geqq l$. *Then for* $i \geqq j \geqq l$

$$\|x_i - x_j\| \leqq A \prod_{k=l+1}^{j} (1+h_k)^{-1} + B \sum_{k=j+1}^{i} h_k + \sum_{k=l+1}^{i} h_k \|z_k\| + \sum_{k=l+1}^{j} h_k \|z_k\|. \tag{2.7}$$

PROOF. Set $a_{i,j} = \|x_i - x_j\|$. From the accretiveness of B and (2.1) we obtain (see, e.g. [2] lemma 1.7 and remark 4.1) the following recursion relation:

$$a_{i,j} \leqq \frac{h_i + h_j}{h_i + h_j + h_i h_j} \left(\frac{h_j}{h_i + h_j} (a_{i-1,j} + h_i \|z_i\|) + \frac{h_i}{h_i + h_j} (a_{i,j-1} + h_j \|z_j\|) \right). \tag{2.8}$$

By assumption, (2.7) holds if $j = l$ and clearly it holds if $i = j$. Thus it suffices to show that if $i > j \geqq l$ and the desired estimate (2.7) holds for $a_{i-1,j}$ and $a_{i,j-1}$ then it holds for $a_{i,j}$. However, this follows readily from (2.8).

LEMMA 3. *Let* (2.1) *hold and* $\Sigma_{k=1}^{\infty} h_k \|z_k\| < \infty$. *Then* $\lim_{k\to\infty} x_k$ *exists.*

PROOF. There are two cases. First we assume $\Sigma_{k=0}^{\infty} h_k = \infty$. By Lemma 1

$$\|x_i - x_l\| \leqq \|x_l + y_l\| + \sum_{k=l+1}^{i} h_k \|z_k\| \quad \text{for} \quad i \geqq l,$$

so by Lemma 2 (with $A = \|x_l + y_l\|$, $B = 0$)

$$\|x_i - x_j\| \leqq \|x_l + y_l\| \prod_{k=l+1}^{j} (1+h_k)^{-1} + \sum_{k=l+1}^{i} h_k \|z_k\| + \sum_{k=l+1}^{j} h_j \|z_j\| \tag{2.9}$$

for $i \geqq j \geqq l$. Since $\Sigma_{k=1}^{\infty} h_k = \infty$, $\Pi_{k=l+1}^{\infty}(1+h_k)^{-1} = 0$. Thus by (2.9)

$$\limsup_{i,j\to\infty} \|x_i - x_j\| \leqq 2 \sum_{l+1}^{\infty} h_k \|z_k\| \tag{2.10}$$

for all l. Letting $l \to \infty$ we find that $\{x_k\}$ is a Cauchy sequence and hence convergent.

The other possibility is that $\Sigma_{k=0}^{\infty} h_k < \infty$. This time we use the assertion

$$\|x_i - x_l\| \leqq \|x_l + y_l\| \sum_{k=l+1}^{i} h_k + \sum_{k=l+1}^{i} h_k \|z_k\| \qquad i \geqq l,$$

from Lemma 1. Now Lemma 2 (with $A = 0$, $B = \|x_l + y_l\|$) implies

$$\|x_i - x_j\| \leqq \|x_l + y_l\| \sum_{k=j+1}^{i} h_k + \sum_{k=l+1}^{i} h_k \|z_k\| + \sum_{k=l+1}^{j} h_k \|z_k\|.$$

Letting $i, j \to \infty$ we again find (2.10) (since $\Sigma h_k < \infty$), and $\{x_k\}$ is Cauchy as before.

PROOF OF THEOREM 1. Part (a) is an immediate consequence of Lemma 3. To obtain part (b) let $\lim_{k\to\infty} x_k = x_\infty$ and $y \in Bx$. Rewriting (2.1) we have

$$y_{k+1} - y = h_{k+1}^{-1}(x_k - x_{k+1} + h_{k+1}(z_{k+1} - x_{k+1} - y)). \tag{2.11}$$

Since B is accretive

$$\| x_{k+1} - x \| \leqq \| x_{k+1} - x + \mu (y_{k+1} - y)\| \quad \text{for} \quad \mu > 0. \tag{2.12}$$

Substitute (2.11) in (2.12), take $\mu = \lambda h_{k+1}(\lambda + h_{k+1} + \lambda h_{k+1})^{-1}$ and multiply by $\lambda^{-1}(\lambda + h_{k+1} + \lambda h_{k+1})$ to find:

$$\begin{aligned}(1 + h_{k+1} + \lambda^{-1} h_{k+1})\| x_{k+1} - x \| &\leqq \| x - x_k - h_{k+1} z_{k+1} + h_{k+1}(x + y) \\ &+ \lambda^{-1} h_{k+1}(x - x_{k+1})\| \leqq \| x - x_k \| + h_{k+1}\| z_{k+1}\| + \lambda^{-1} h_{k+1}\|(x - x_{k+1}) + \lambda (x + y)\|\end{aligned}$$

and therefore

$$\begin{aligned}(1 + h_{k+1})\| x - x_{k+1}\| \leqq \| x - x_k \| &+ \lambda^{-1} h_{k+1}(\| x - x_{k+1} + \lambda (x + y)\| \\ &- \| x - x_{k+1}\|) + h_{k+1}\| z_{k+1}\|.\end{aligned} \tag{2.13}$$

Iterating this inequality from $k = l$ to $k = i - 1$ yields

$$\begin{aligned}\| x - x_i \| \leqq \| x - x_l \| &\prod_{j=l+1}^{i} (1 + h_j)^{-1} \\ &+ \lambda^{-1} \sum_{j=l+1}^{i} \left(\prod_{m=j}^{i} (1 + h_m)^{-1}\right) h_j(\| x - x_j + \lambda (x + y)\| - \| x - x_j \|) \\ &+ \sum_{j=l+1}^{i} \left(\prod_{m=j}^{i} (1 + h_m)^{-1}\right) h_j \| z_j \|.\end{aligned} \tag{2.14}$$

Since $\Sigma_{j=1}^{\infty} h_j = \infty$, the first term on the right of (2.14) tends to zero as $i \to \infty$. The third term on the right also tends to zero as $i \to \infty$ by the dominated convergence theorem (each term individually tends to 0 and $\{h_j \| z_j \|\}$ is a dominating summable sequence). Finally, by (2.6) and

$$\lim_{j\to\infty} (\| x - x_j + \lambda (x + y)\| - \| x - x_j \|) = \| x - x_\infty + \lambda (x + y)\| - \| x - x_\infty \|$$

we obtain upon letting $i \to \infty$ in (2.14) that

$$\| x - x_\infty \| \leqq \lambda^{-1}(\| x - x_\infty + \lambda (x + y)\| - \| x - x_\infty \|)$$

or, after rearranging,

$$(1+\lambda)\|x - x_\infty\| \leqq \|x - x_\infty + \lambda(x+y)\|$$

which is the desired inequality.

PROOF OF THEOREM 3. We begin by showing that if B is accretive and satisfies (R) then there is an admissible sequence for $0 \in R(I+B)$.

For $x \in \overline{D(B)}$ and $\varepsilon > 0$ let $\Lambda(x,\varepsilon)$ be the set of those numbers $\lambda > 0$ for which there exists x_λ and $y_\lambda \in Bx_\lambda$ such that

$$\|x_\lambda + \lambda(x_\lambda + y_\lambda) - x\| < \chi(\lambda)\varepsilon$$

where $\chi(\lambda) = \min(1,\lambda)$ for $\lambda \in (0,\infty]$. $\Lambda(x,\varepsilon)$ is nonempty by condition (R), and we define $\lambda(x,\varepsilon) = \sup \Lambda(x,\varepsilon)$. Let $x_0 \in \overline{D(B)}$ be arbitrary and suppose $x_1, x_2, \cdots, x_{k-1}$ and $h_1, h_2, \cdots, h_{k-1}$ have been chosen. Set

$$\varepsilon_k = \exp\left(-\sum_{j=1}^{k-1} h_j - 1\right) \tag{2.15}$$

and choose $h_k > 0$, $x_k, y_k \in Bx_k$ so that

$$\begin{cases} \chi(\tfrac{1}{2}\lambda(x_{k-1},\varepsilon_k)) \leqq h_k < \infty \\ \|x_k - x_{k-1} + h_k(x_k + y_k)\| < \chi(h_k)\varepsilon_k. \end{cases} \tag{2.16}$$

In this way we get infinite sequences $\{h_k\}$, $\{x_k\}$, $\{y_k\}$. Now by (2.16), (2.15) and with z_k as in (2.1),

$$\sum_{k=1}^{\infty} h_k \|z_k\| < \sum_{k=1}^{\infty} \chi(h_k)\varepsilon_k \leqq \int_0^\infty e^{-s} ds < \infty. \tag{2.17}$$

Thus, if $\Sigma_1^\infty h_k = \infty$, $\{x_k\}$ is admissible and we are done. Let $\sigma_k = h_1 + \cdots + h_k$. Assuming $\lim_{k\to\infty} \sigma_k = \sigma_\infty < \infty$ we will reach a contradiction by use of a now standard idea of Nagumo [6], and thus complete the proof of (a). By Lemma 3, $x_\infty = \lim_{k\to\infty} x_k$ exists. By (R) there exists $\lambda_0 \in (0,\infty)$, $\bar{x}, \bar{y} \in B\bar{x}$ such that

$$\|\bar{x} - x_\infty + \lambda_0(\bar{y} + \bar{x})\| < e^{-(\sigma_\infty + 1)}\chi(\lambda_0).$$

Then

$$\|\bar{x} - x_{k-1} + \lambda_0(\bar{y} + \bar{x})\| < e^{-(\sigma_{k-1}+1)}\chi(\lambda_0) = \chi(\lambda_0)\varepsilon_k$$

for all k large enough. Hence, by (2.16), $2h_k \geqq \chi(\lambda_0)$ for large k, contradicting $\sigma_k = h_1 + \cdots + h_k \to \sigma_\infty < \infty$.

To prove part (b), we use (R) to assert the existence of sequences $\lambda_i > 0$, $u_i \in D(B)$, $v_i \in Bu_i$ such that

$$\left\| \frac{u_i - x_\infty}{\lambda_i} + (u_i + v_i) \right\| \to 0. \tag{2.18}$$

By Theorem 1(b) we also have

$$(1+\lambda_i)\left\| \frac{u_i - x_\infty}{\lambda_i} \right\| \leqq \left\| \frac{u_i - x_\infty}{\lambda_i} + (u_i + v_i) \right\|. \tag{2.19}$$

Combining (2.18) and (2.19) we conclude $\lambda_i^{-1}(u_i - x_\infty) \to 0$ as well as $u_i \to x_\infty$ and subsequently, from (2.18), $u_i + v_i \to 0$. Thus $0 \in \overline{R(I+B)}$ and $0 \in R(I+B)$ if B is closed.

PROOF OF THEOREM 2. We first show that (1.3), (1.4) defines an admissible sequence. With $y_k = Bx_k$ we have, by (1.3), (1.4)

$$\begin{aligned}\sum_{k=0}^{\infty} \| x_{k+1} - x_k + h_{k+1}(x_{k+1} + y_{k+1}) \| &= \sum_{k=0}^{\infty} h_{k+1} \| Bx_{k+1} - Bx_k \| \\ &\leqq \sum_{k=0}^{\infty} h_{k+1} \exp\left(- \sum_{j=1}^{k} h_j - 1 \right) \leqq 1.\end{aligned}$$

Thus it is sufficient to show $\Sigma_{k=0}^{\infty} h_j = \infty$. Suppose $\sigma_k = h_1 + \cdots + h_k \to \sigma_\infty < \infty$. Then x_k converges to a limit x_∞ by Lemma 3. Since B is continuous there is an integer n such that

$$\left\| B\left(\frac{2^n}{1+2^n} x_\infty - \frac{1}{1+2^n} Bx_\infty \right) - Bx_\infty \right\| < \exp(-\sigma_\infty - 1).$$

Then also

$$\left\| B\left(\frac{2^n}{1+2^n} x_k - \frac{1}{1+2^n} Bx_k \right) - Bx_k \right\| < \exp(-(h_1 + \cdots + h_k) - 1)$$

for large k and therefore $n_k \leqq n$ for large k. But then $\sigma_k = h_1 + \cdots + h_k \to \infty$ by (1.3), a contradiction.

Let $\{x_k\}$ be an admissible sequence for $0 \in R(I+B)$ and (2.1) hold with $\Sigma_{k=1}^{\infty} h_k \| z_k \| < \infty$. Then $x_k \to x_\infty$ and summing (2.1) from $k = m$ to $k = n-1$ yields

$$\sum_{j=m+1}^{n} \frac{h_j}{\sigma_n - \sigma_m}(x_j + y_j) = \frac{x_m - x_n}{\sigma_n - \sigma_m} + \frac{1}{\sigma_n - \sigma_m} \sum_{j=m+1}^{n} h_k z_k,$$

where $\sigma_j = h_1 + \cdots + h_j$. The right hand side above tends to zero as $n, m \to \infty$

subject to $\sigma_n - \sigma_m \geqq 1$, while the left hand side consists of convex combinations of $x_j + y_j$. Thus if $x_j + y_j$ has a limit as $j \to \infty$, it must be zero. If B is continuous and $D(B)$ is closed, $x_\infty \in D(B)$ and $x_j + Bx_j \to x_\infty + Bx_\infty$, proving (b).

PROOF OF PROPOSITION 1. Let $0 \in \overline{R(I+B)}$. Let $\{\alpha_k\}$ be an arbitrary summable sequence of positive numbers and $\{h_k\}$ a sequence of positive numbers satisfying $\Sigma_{k=1}^{\infty} h_k = \infty$, $\Sigma_{k=1}^{\infty} \alpha_k h_k < \infty$. Choose $x_k \in D(B)$, $y_k \in Bx_k$ such that $\|x_k + y_k\| < \alpha_k$. Then we claim $\{x_k\}$ is an admissible sequence for $0 \in R(I+B)$ and we may use $\{h_k\}$ in (1.2). Indeed

$$\|x_{k+1} - x_k\| \leqq \|x_{k+1} - x_k + y_{k+1} - y_k\| \leqq \|x_{k+1} + y_{k+1}\| + \|x_k + y_k\| \leqq \alpha_{k+1} + \alpha_k$$

so

$$\sum_{k=1}^{\infty} \|x_{k+1} - x_k + h_{k+1}(x_{k+1} + y_{k+1})\| \leqq \sum_{k=1}^{\infty} \|x_{k+1} - x_k\| + h_{k+1}\|x_{k+1} + y_{k+1}\|$$

$$\leqq 2\sum_{k=1}^{\infty} \alpha_k + \sum_{k=1} \alpha_k h_k < \infty.$$

Thus $\{x_k\}$ is admissible.

In the proof of Theorem 2 we showed that certain convex combinations of $x_j + y_j$ converged to 0 if (1.2) holds, so 0 is in the closed convex hull of $R(I+B)$ if there is an admissible sequence. This completes the proof.

REMARKS. From the proof of Proposition 1 we see that if $0 \in \overline{R(I+B)}$ then there exist admissible sequences with arbitrary associated sequences $\{h_k\}$ satisfying $\Sigma h_k = \infty$. It is worth noting that if (1.2) holds and $\inf_k h_k > 0$ then $0 \in \overline{R(I+B)}$ for

$$h_k(x_k + y_k) = (x_{k-1} - x_k + h_{k+1}(x_{k+1} + y_{k+1})) - (x_{k-1} - x_k)$$

and the first term on the right tends to 0 by (3) (ii) and the second does by Theorem 1, so $x_k + y_k \to 0$. One can also show that $x_k + y_k \to 0$ if $\Sigma_{k=1}^{\infty} \|z_k\| < \infty$.

3. Applications

In this section we deduce two known general theorems on accretive sets in Banach space as simple consequences of our results. We note that the previously known proofs of these theorems were all dependent on the existence of a solution to the initial value problem (1).

We start by introducing the following conditions:

$$(\mathrm{R}_1) \qquad \liminf_{\lambda \to 0^+} \lambda^{-1} \operatorname{dist}(R(I+\lambda A); x - \lambda y) = 0 \quad \forall x \in \overline{D(A)}, \qquad y \in X.$$

Note that it is sufficient for (R_1) to hold only for every y in a dense subset of X in order for it to hold for every $y \in X$.

LEMMA 4. *Let A be accretive and satisfy* (R_1). *If $P: \overline{D(A)} \subset X \to X$ is continuous, then $A + P$ satisfies* (R_1).

PROOF. If A satisfies (R_1) then for every $y \in X$ there are sequences $\lambda_i \to 0$ and $y_i \in Ax_i$ such that

$$\lambda_i^{-1}(x_i + \lambda_i y_i - (x - \lambda_i y - \lambda_i Px)) \to 0 \quad \text{as} \quad i \to \infty. \tag{3.1}$$

But

$$\lambda_i^{-1}(x_i + \lambda_i y_i + \lambda_i Px_i - (x - \lambda_i y)) = \lambda_i^{-1}(x_i + \lambda_i y_i - (x - \lambda_i y - \lambda_i Px)) + Px_i - Px.$$

In order to prove that $A + P$ satisfies (R_1) it is therefore sufficient to show that $x_i \to x$ as $i \to \infty$. from the accretiveness of A we have

$$\|x_i - u\| \leqq \|x_i - u + \lambda_i(y_i - v)\| \quad \text{for} \quad v \in Au$$

and from (3.1) we then have

$$\limsup_{i \to \infty} \|x_i - u\| \leqq \limsup_{i \to \infty} \|x - u - \lambda_i(y + Px + v)\| = \|x - u\|$$

for all $u \in D(A)$ and thus $x_i \to x$.

Let $z \in X$ be arbitrary. Taking $P = I - z$ in the previous lemma, it follows that $A_z = A + I - z$ satisfies condition (R_1) and hence also (R) of Section 1 and therefore by Theorem 3 we have:

THEOREM (Y. Kobayashi [3]). *Let A be accretive and satisfy* (R_1). *Then $\bar{A}$ (the closure of A) is m-accretive.*

We conclude with the following general perturbation theorem.

THEOREM (Y. Kobayashi [3]). *Let A be accretive and $P: \overline{D(A)} \subset X \to X$ be continuous. If $A + P$ is accretive, then it is m-accretive if and only if A is m-accretive.*

PROOF. Let A be m-accretive then A is closed and satisfies (R_1). Therefore, by Lemma 4, $A + P$ satisfies (R_1). Since, by the continuity of P, $A + P$ is closed and by assumption it is accretive, it follows from the previous theorem that $A + P$ is m-accretive. If $A + P$ is m-accretive one has that $A = (A + P) - P$ is as well by the above.

The last theorem is a considerable generalization of the theorem of R. Martin which was stated in the introduction. Indeed Martin's theorem is obtained by taking $A \equiv 0$ on X. G. Webb [9] proved the above perturbation result assuming that A was linear m-accretive with $\overline{D(A)} = X$. Subsequently V. Barbu [1] generalized Webb's result to the case where A was a general m-accretive operator and $P: X \to X$ was continuous. Finally Y. Kobayashi [3] proved the above theorem.

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Mathematics Research Center
University of Wisconsin — Madison